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The Effect of Thermal Radiation and Variable Viscosity Parameters on a Fluid Flow Down Along an Inclined Plane With Free Surface

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ABSTRACT

This paper investigates the effects of thermal radiation and variable viscosity flow down along an inclined plane with boundary conditions at free surface. The major problem includes internal heat generation, increase or decrease in temperature, and other thermo physical properties. The thermo physical properties include Grashf number, Nusselt number, Viscosity and Solar radiation parameter. The problems created have not been examined. Thus, this work examined the effect of temperature and velocity profiles on the various values of coefficient of viscosity, also the effects of solar radiation parameter on the major property of the fluid flow down along an inclined plane.

The partial differential equations for the problem are continuity, momentum and energy equations. These are non- linear dimensionless equations governing the fluid flow down the inclined plane using integration method. The equations for the fluid flow, temperature and velocity of the problem are reduced to their final forms using perturbation method. Analytical expressions are employed to obtain the value of the velocity and temperature profiles in terms of parameters under the considerations in the flow field. The parameters are the major factors influencing the properties of the fluid flow down along an inclined plane.

Hence, the viscosity of the fluid increases as the velocity of the fluid decreases while increase in the solar radiation parameter increases velocity of the fluid. Also the quantities of radiant energy absorbed by the fluid flow bring changes in the temperature of the fluid. Increase in Nusselt decreases the velocity of the fluid. Grashof number increases while the temperature of the fluid also decreases.

In conclusion, viscosity of the fluid decreases with an increase in temperature due to cohesion and molecular momentum exchange between fluid layer and the parameters are found to have a significant effect over the velocity and temperature profiles of the fluid flow down an inclined plane at free surface. It will also useful for the industries in the production of the various fluids (liquid or gas) such as vegetable oil, palm oil and steam generation along an inclined plane and so on.

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الخلاصة

وتحقق هذه الورقة في آثار الإشعاع الحراري وتدفق اللزوجة المتغيرة إلى الأسفل على طول مستوى مائل مع ظروف حدودية على سطح حر. وتشمل المشكلة الرئيسية توليد الحرارة الداخلية ، وزيادة درجة الحرارة أو خفضها ، وغير ذلك من الخصائص الفيزيائية الحرارية. وتشمل الخواص الفيزيائية الحرارية رقم Grashof، ورقم Nusselt ، وعامل اللزوجة والإشعاع الشمسي. ولم تبحث المشاكل الناشئة. وبالتالي ، فإن هذا العمل درس تأثير درجة الحرارة و سرعة لمحات عن مختلف قيم معامل اللزوجة أيضا آثار الإشعاع الشمسي المعلمة على الممتلكات الرئيسية من السوائل تتدفق على طول المنحدر.
المعادلات التفاضلية الجزئية للمشكلة هي الاستمرارية والزخم ومعادلات الطاقة. هذه معادلات بلا أبعاد غير خطية تحكم تدفق السائل أسفل المستوى المميل باستخدام طريقة التكامل. معادلات تدفق السوائل ودرجة الحرارة وسرعة المشكلة يتم تخفيضها إلى أشكالها النهائية باستخدام طريقة الاضطرابات. وتستخدم

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التعبيرات التحليلية للحصول على قيمة ملامح السرعة ودرجة الحرارة من حيث البارامترات في إطار الاعتبارات في مجال التدفق. البارامترات هي العوامل الرئيسية التي تؤثر على خصائص تدفق السائل إلى أسفل على طول مستوى مائل

وبالتالي ، فإن لزوجة السائل تزداد مع انخفاض سرعة السائل في حين أن الزيادة في معامل الإشعاع الشمسي تزيد سرعة السائل. كما أن كميات الطاقة المشعة التي يمتصها تدفق السوائل تحدث تغيرات في درجة حرارة السائل. الزيادة في (نوسيلت) تقلل من سرعة السائل يزيد عدد الجراشوف بينما تنخفض درجة حرارة السائل

في الختام لزوجة السوائل يتناقص مع زيادة في درجة الحرارة بسبب التماسك الجزيئية زخم التبادل بين السائل طبقة المعلمات وجدت أن يكون لها تأثير كبير على سرعة و درجة الحرارة لمحات من السوائل تتدفق الطائرة تميل على السطح. كما أنه سيكون مفيدا للصناعات في إنتاج السوائل المختلفة (السائل أو الغاز) مثل الزيت النباتي وزيت النخيل وتوليد البخار على طول مستوى مائل وما إلى ذلك

1. INTRODUCTION

Fluid mechanics is one of the core applied Mathematics which deals with the behavior of fluid under the conditions of rest or motion; Alhama, *et al* (2007). The discussion is built around the properties of the fluid flow down along an inclined plane with boundary conditions at the free surface; the flow of liquid is always that both the pressure and the shear stress are zero everywhere. Liquid thin flows in conduits or open channels are of interest in science, engineering, and everyday life, Asibor, *et al* (2017), Aziz, A. (2009). The study of the temperature-dependent (thermal conductivity) and fluid viscosity of a thin liquid film along an inclined plane with a free surface are important because of their wide applications in several industries, Costa, *et al* (2003). Examples may be found in the melting of rods, aluminum during the recycling processes, continuous flow of liquid in beverage industries, water cooperation, painting industries, and so on, Christos, *et al* (2016)

Dianchen, *et al* (2019) investigated the effects of variable viscosity and variable thermal conductivity on heat transfer from moving surfaces with radiation. In their work, the fluid viscosity also varies as an inverse linear function of temperature, and the thermal conductivity varies as a linear function of temperature, Disu, *et al* (2009). The effect of convective heat transfer is extremely important in understanding the flow structure of many fluids used in industrial and natural applications, Eegunjobi, *et al* (2013). The present paper is aimed at investigating the effect of convective heat transfer on the flow of a viscous fluid with exponential temperature-dependent viscosity, down an inclined plane with a free surface. Recently, Elbarbary, *et al* (2004) have studied the effect of variable thermal conductivity in a non-isothermal sheet stretching through power-law fluids. Similar studies for the viscoelastic fluids have been reported by Elbashbeshy, *et al* (2004). Both studies revealed that the effect of variable thermal conductivity is to increase the shear stress. The thickness of the thermal boundary layer relative to the velocity boundary layer depends on the Prandtl number which by its definition varies directly with the fluid viscosity and inversely with the thermal conductivity of the fluid, Elbashbeshy, *et al* (2000). As the viscosity and the thermal conductivity vary with temperature so does the Prandtl number. Despite this fact, all of the afore-mentioned studies treated the Prandtl number as a constant, Hao, *et al* (2010). The use of a constant Prandtl number within the boundary layer when the fluid properties are temperature-dependent introduces errors in the computed results, Hazarika, *et al* (2015).

Recently, Isaac, L.A. (2017) studied the hydromagnetic flow of a Newtonian fluid over an inclined plate with variable viscosity whereas Gabriella, *et al.* (2012) studied the flow of a micropolar fluid with variable viscosity over a permeable stretching sheet, Koríko, *et al.* (2012). Both studies confirmed that for the accurate prediction of the thermal characteristics of variable viscosity fluid flows, the Prandtl number must be treated as a variable rather than a constant, Makinde, O. D. (2006). These studies, however, assumed the thermal conductivity to be a constant. In another study, Makinde, *et al.* (2005) investigated the effects of variable electric conductivity and non-uniform heat source (or sink) on convective micropolar fluid flow along with an inclined flat plate with constant surface temperature, Mohammed, *et al*. (2010). They found that the skin-friction coefficient and Nusselt number are higher for the case of constant fluid electric conductivity than for the case of variable fluid electric conductivity, Muhim, C (2018). In their model, they treated fluid viscosity and thermal conductivity to be constants.

Myers, *et al.* (2006) solved the laminar thermal boundary layer flow over a flat plate with a convective surface boundary condition by applying the similarity variable and presented the Biot number, Olanrewaju, *et al*. (2011). The study of convective heat transfer in a viscous incompressible fluid over flat plate has received considerable attention due to its application in processes involving high temperatures such as gas turbines, nuclear, power plants, and thermal storage, Prasad, *et al.* (2009). The problem of fluid flow over a horizontal, stationary flat plate in a uniform free stream was first solved by Prasad, *et al.* (2010). This was done by transforming the governing partial differential equations into ordinary differential equations by introducing a new independent variable called the similarity variable, Rahman, *et al.* (2010). The similarity variable has been applied to solve the thermal boundary layer for the constant surface temperature at the plate on the heat transfer characteristics, Rahman, *et al.* (2009).

Rajput, E. R. (2013); The steady laminar boundary layer flow of a non-Newtonian fluid over an impermeable flat plate with convective boundary condition was investigated by Sandy, *et al.* (2010) the power-law index of the fluid was considered. Given the above paper, the effect of some fluid properties such as temperature, and viscosity related to variable numbers that are Biot number, Brickman number) has been investigated, Saouli, *et al.* (2004). Therefore, the present paper is aiming at investigating the effect of thermal radiation and variable fluid viscosity along an inclined plane with the free surface, Srinivasacharya, et al. (2019) and Tshehla, M.S. (2013) by considering Grash of number, angle of inclination, Nusselt number, the flow of a viscous fluid and thermal conductivity with exponential temperature-dependent viscosity, down an inclined plane with a free surface, Usman, M. A. and Onitilo, S. A. (2013).

2. Problem Formulation

Consider a steady boundary laminar flow and heat transfer of a viscous incompressible fluid down an infinite inclined plane. Let L be the direction of the fluid flow along the x-axis as the main flow along an inclined plane of the sheet and h be the width in the y-direction along the y-axis. The word infinite implies that the length of the plane is greater than the L. Hence, the flow may be treated as two-dimensional ($\frac{\partial u}{\partial z}$ = 0). For the flow is steady, the flow variables are independent of time ($\frac{\partial u}{\partial t}$ = 0). With a free surface, the energy coming off the sun reaches the fluid in the form of electromagnetic waves after experiencing considerable interaction with the atmosphere. Hence, Solar radiation is introduced (R_s) which varies directly with the quantity of heat gained by the fluid, Gabriella, et al. (2012).

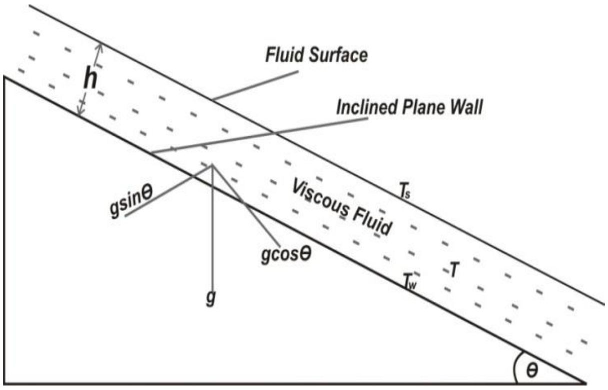


Fig.1. Geometry of the problem. Source: Researcher (2019).

where θ is an inclination angle, T_s is the surface temperature, T_l is the lower temperature, T is the temperature of the fluid, h represents the width along the y-axis and g is the acceleration due to gravity.

Under the foregoing assumptions and invoking the usual boundary layer approximation, the governing equations for the Continuity, Momentum and Energy equations of a viscous fluid in the presence of Variable fluid properties (fluid viscosity and thermal conductivity) take the following form.

3. Continuity Equation

$$\nabla . u = 0 \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

3.1 Momentum Equation

The momentum equation for the fluid flow for x-axis and y-axis respectively are written in terms of components below:

X – Component, we have

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g\beta(T - T_l)\sin\theta$$
$$+2\left(\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial y \partial x}\right. \quad (3)$$

Y − Component, we have;

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g\beta(T - T_l)\cos\theta$$
$$+ 2\left(\mu\frac{\partial^2 v}{\partial y^2}\right) + \mu\frac{\partial^2 u}{\partial y \partial x} + \mu\frac{\partial^2 v}{\partial x^2} \quad (4)$$

3.2 Energy Equation

In optically thin limit radiation, the radiation flux vector is given

$$Q_r = -4\sigma K\frac{\partial T^4}{\partial y} \quad (5)$$

The conservation of energy equation is given by

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)\right] - Q_r \quad (6)$$

non- dimensionless parameters and variable employed for this study are as follows;

$$x' = \frac{x}{L},\ y' = \frac{y}{H},\ u' = \frac{u}{U},\ v' = \frac{vL}{hU},\ t' = \frac{Ut}{L}\ \ \mu' = \mu_0\mu'$$
$$p = P = \frac{\mu 0uL}{h2}P'\ \ T' = \frac{T - T_l}{T_s - T_l} \quad (7)$$

To simplify notation, the primes are omitted from now on. Since the film is thin liquid, the aspect ratio ε = h/L≪1. Using the scaled parameters, (3.2 – 3.6) now becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (8)$$

$$\varepsilon^2 Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + 1 + 2\varepsilon^2 \quad (9)$$
$$\mu\left(\frac{\partial^2 u}{\partial x^2}\right) + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) + \mu\varepsilon^2\left(\frac{\partial^2 v}{\partial y \partial x}\right)$$

$$\varepsilon^{2\,Re4}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + Gr\varepsilon\cot\theta$$
$$+2\varepsilon^2\mu\left(\frac{\partial^2 v}{\partial y^2}\right) + \varepsilon^2\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \varepsilon^2\frac{\partial v}{\partial x}\right)\right] \quad (10)$$

$$\varepsilon^2 Pe\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\right) = \varepsilon^2 k\frac{\partial 2T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2}$$
$$+ NuEc\left[\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right] + Gr\,\mu\left[2\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right.$$
$$\left. +2\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right] - \frac{\partial Q_r}{\partial y} \quad (11)$$

where $Gr = g\beta\frac{k(T - T_s)L^3}{U^2}$, $U = \frac{\rho g H^2 \sin\theta}{\mu_0}$ and

$$Nu = \frac{hL}{K}\ \ Pe = \frac{\rho c_p UL}{k}\ \ P = \frac{\mu_0 UL}{h^2} \quad (11a)$$

3.3 Boundary Conditions

(i) At y=0, the temperature at the lower surface is constant:

u (0) = 0, v (0) = 0, T (0) = 0, at y=0 and

(ii) At. y =1, $\left(\frac{\partial u}{\partial y}\right)|_{y=1,} = 0$ $\left(\frac{\partial T}{\partial y}\right)|_{y=1} = Nu(T - 1)$ (12)

Where $Nu = \frac{hL}{K}$ is the Nusselt number and represent the ratio of heat transfer between a moving fluid and a solid body. The conditions are similar to conditions in Alhama and Zueco (2007) and Makinde (2006).

In an Optically thin limit, the radiant absorption is expressed as the thermal radiation flux:

$$Q_r = -4\sigma K\frac{\partial T^4}{\partial y} \quad (13)$$

Applying Taylor series about T_∞ and neglecting higher-order terms to give:

$$T^{`4} = (4T_\infty^3 T - 3T_\infty^4) \quad (14)$$
$$T^{`4} = T_\infty^4 - T^{`4} \quad (15)$$

Substituting equation (15) into equation (13) we have:

$$\frac{\partial Q_r}{\partial y^2} = 16\sigma K (T_\infty^{3}) \frac{\partial^2 T}{\partial y^2} \quad (16)$$

All the values of the parameters such as Peclet number, Renold number and reduced quantity PrEc are all assumed to be very small and neglected. The Grashof number may be very close to unity and must be retained in the equations (8 – 11). Using this condition, the equations are reduced to their final form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (17)$$

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}[\mu(\frac{\partial u}{\partial y}) + 1 = 0 \quad (18)$$

$$-\frac{\partial p}{\partial y} = 0 \quad (19)$$

$$\frac{\partial^2 T}{\partial y2} = -\text{Gr}\,\mu\,(\frac{\partial u}{\partial y})^{2} - 16\sigma K(T_\infty^{3})\frac{\partial^2 T}{\partial y^2} \quad (20)$$

Let $R_s = 16\sigma K(T_\infty^{3})$,then, equation (20) is reduced to

$$\frac{\partial^2 T}{\partial y2} = \frac{1}{1+R_s}(-\text{Gr}\,\mu\,(\frac{\partial u}{\partial y})^{2}) \quad (21)$$

3.4 Variable Viscosity Analysis

$$\mu = \mu_0\, e^{-\phi(T - T_l)} \quad (22)$$

Where μ_0 is the reference viscosity at the reference temperature T_0 and ϕ is the coefficient of viscosity with temperature Costa and Macedonio(2003).

Using non- dimensional parameters from equation (7), then equation (22) becomes;

$$\mu = e^{-\phi \Delta T T 1} \quad (23)$$

$$\text{Let } @ = \phi\, \nabla T^{l} \quad (24)$$

$$\mu = e^{-@T} \quad (25)$$

Equation (25) is known as Naheme's exponential law, see Myers and Et-al, @ is a constant called coefficient of viscosity variation.

$$\text{Let } \frac{\partial p}{\partial x} = 0 \quad (26)$$

Substituting equations (19) and (20) into equation (13), we have:

$$\frac{\partial}{\partial y}[\mu(\frac{\partial u}{\partial y})] + 1 = 0 \quad (26a)$$

Integrating equation (18) with respect to y we have:

$$\mu(\frac{\partial u}{\partial y}) = A - y \quad (27)$$

Dividing both sides of equation (22) by μ we have:

$$\frac{\partial u}{\partial y} = (A - y)\,\mu^{-1} \quad (28)$$

Substituting equations (25) into equation (28) we have:

$$\frac{\partial u}{\partial y} = (A - y)\, e^{@T} \quad (29)$$

Substituting equations (29) into equation (21) we have:

$$\frac{\partial^2 T}{\partial y2} = \frac{1}{1+R_s}(-\text{Gr}\,\mu\,(\frac{\partial u}{\partial y})^{2})\, e^{2@T} \quad (30)$$

Substituting equation (29) into equation (30) we have:

$$\frac{\partial^2 T}{\partial y2} = \frac{1}{1+R_s}(-\text{Gr}\, e^{-@T}\,(\frac{\partial u}{\partial y})^{2})\, e^{2@T} \quad (31)$$

$$\frac{\partial^2 T}{\partial y2} = \frac{1}{1+R_s}(-\text{Gr}\,(A - y) e^{@T}) \quad (31a)$$

Therefore, equation (31) represents the temperature profile and equation (29) represents velocity profile.

4. Method of Solution

Equation (29) cannot be integrated further to determine velocity, since it involves unknown temperature T. In order to solve equations (31) and (29) subject to boundary conditions, we assume that the variation in the fluid viscosity is small ($0 < \varphi \ll 1$) and seek an asymptotic solution for the fluid velocity and temperature of the form:

$$u = u_0 + \varphi\, u_1 \quad (32)$$

$$T = T_0 + \varphi T_1 \quad (33)$$

A similar expression for equations (32- 33) can be obtained in Tshela, (2013)

Substituting for T in equation (31), we have:

$$\frac{\partial^2 (T0 + @\,T1)}{\partial y^2} = \frac{1}{1+R_s}(-\text{Gr}\,(A\, e^{@[T0(y) + @\,T1(x,y)]}) \quad (34)$$

Using Taylor series expansion, we have:

$$\frac{\partial^2 T_0}{\partial y2} + \varphi\frac{\partial^2 T_1}{\partial y^2} = -\frac{1}{1+R_s}\text{Gr}\,(A - y)^2\,[+\frac{@[T0(y) + @\,T1(x,y)]^2}{2!} + \cdots \quad (35a)$$

$$\frac{\partial^2 T_0}{\partial y^2} + \varphi\frac{\partial^2 T_1}{\partial y^2} = (\frac{1}{1+R_s})-\text{Gr}[(A-y)^2 + \varphi T_0(A-y)^2] \quad (35b)$$

The leading order of φ, in the equation(35b), is now reduced to:

$$\varphi^0;\ (A-y)^2 \ \text{and}\ \varphi^{l};\ T_0\,(A-y)^2 \quad (35c)$$

$$\frac{\partial^2 T_0}{\partial y^2} = (\frac{1}{1+R_s})(-\text{Gr}\,(A-y)^2) \quad (36)$$

$$\frac{\partial^2 T_1}{\partial y^2} = (\frac{1}{1+R_s})(-\text{Gr}\,T_0\,(A-y)^2) \quad (37a)$$

Equation (36) and (37a) are solved subject to the boundary conditions thus; (37b)

$$T_0 = T_1 = 0,\ \text{at } y = 0,\ [(\frac{\partial T_0}{\partial y})]|_{y=1} = -\,Nu\,(T_0 - 1),$$

$$[(\frac{\partial T_0}{\partial y})]|_{y=1} = NuT_1 \quad (38)$$

Transform equation (36) by letting $B = (\frac{1}{1+R_s})$ to give:

$$\frac{\partial^2 T_0}{\partial y^2} = -B\,(A - y)^2 \quad (39)$$

Integrating equation (39i) with respect to y and by applying the boundary conditions:

$$(\frac{\partial T_0}{\partial y}) = \frac{B(A-y)^3}{3} + C \quad (40)$$

$$-Nu(T_0-1) = \frac{B}{3}(A-1)^3 + C \quad (41)$$

$$C = -Nu(T_0-1) - \frac{B}{3}(A-1)^3 \quad (42)$$

Putting equation (33) into equation (41) to give:

$$\frac{\partial T_0}{\partial y} = \frac{B}{3}(A-y)^3 - \frac{B}{3}(A-1)^3 - Nu(T_0-1) \quad (43)$$

Resolving equation (43) into two difference equations to determine T_0, thus:

$$\frac{\partial T_0}{\partial y} = \frac{B}{3}[(A-y)^3 - (A-1)^3] \quad (44A)$$

$$\frac{\partial T_0}{\partial y} = -\,Nu\,(T_0-1) \quad (44B)$$

Integrating equation (44A), with respect to y we have:

$$T_0 = -\frac{B}{12}(-A^4+6A^2y^2-12A^2y-4Ay^3+12Ay+y^4-4y) + C \quad (45A)$$

Integrating equation (44B), with respect to y we have:

$$T_0 = 1 + De^{-Nuy} \quad (45B)$$

Therefore; add equation (45A) and (45B) to give:

$$T_0 = \frac{B}{12}(-A^4+6A^2y^2-12A^2y-4Ay^3+12Ay+y^4-4y) + (De^{-Nuy}+1) + C \quad (46)$$

Applying initial conditions: $T_0 = 0$ at $y=0$ to determine the values of C thus:

$C = -\,(D + 1)$

Substituting C into equation (46), we have:

$$T_0 = \frac{B}{12}(-A^4+6A^2y^2-12A^2y-4Ay^3+12Ay + y^4 - 4y) + (De^{-Nuy}+1) - (D+1) \quad (47)$$

Transform equation (37) to give:

$$\frac{\partial^2 T_1}{\partial y^2} = -BT_0\,(A-y)^2 \quad (48)$$

Substituting T_0 into equation (48) to give:

$$\frac{\partial^2 T_1}{\partial y^2} = -B[\frac{B}{12}(-A^4+6A^2y^2-12A^2y-4Ay^3+12Ay+y^4-4y) + (De^{-Nuy}+1) - (D+1)]\,(A-y)^2 \quad (49)$$

Equation (35d) is further simplified and Integrated with respect to y to give:

$$\frac{\partial T_1}{\partial y} = -\frac{B^2}{12}(A^7y+5A^6y^2+3A^5y^3+6A^5y^2+6A^4y^2+A^4y+18A^4y^3-\frac{39}{5}A^3y^5+2A^3y+18A^3y^3 - 18A^3y^4+\frac{7}{5}A^2y^6+6A^2y^3+18A^2y^4+6A^2y^5-6Ay^4-6Ay^5) - \frac{B^2}{12}(\frac{y^8}{5}-2y^2) + 3A^2BD(\frac{y}{Nu}e^{-Nuy}+y^2) - 3ABD(\frac{y^2}{Nu}e^{-Nuy}-y^3) + BD(\frac{y^3}{Nu}e^{-Nuy}-y^4) + E \quad (50)$$

Applying initial conditions: at $y = 1, [(\frac{\partial T_0}{\partial y})]|_{y=1} = NuT_1$ into equation (50) to determine E :

$$E = -\frac{B^2}{12}(A^7+5A^6+3A^5+6A^5+6A^4+A^4+18A^4-\frac{39}{5}A^3+2A^3+18A^3-18A^3+7A^2+6A^2+18A^2 + 6A^2-6Ay-6A) - \frac{B^2}{12}(\frac{1}{5}-2) + 3A^2BD(\frac{1}{Nu}e^{-Nu}-1) - 3ABD(\frac{1}{Nu}e^{-Nu}-1) - BD(\frac{1}{Nu}e^{-Nu}+1) - NuT_1 \quad (51)$$

Substituting equation (51) into equation (50) to give:

$$\frac{\partial T_1}{\partial y} = -\frac{B^2}{12}(A^7y+5A^6y^2+3A^5y^3+6A^5y^2+6A^4y^2+A^4y^4+18A^4y^3-\frac{39}{5}A^3y^5+2A^3y^2 +18A^3y^3-18A^3y^4+\frac{7}{5}A^2y^6+6A^2y^3+18A^2y^4+6A^2y^5-6Ay^4-6Ay^5) - \frac{B^2}{12}(\frac{y^8}{5}-2y^2) + 3A^2BD(\frac{y}{Nu}e^{-Nuy}+y^2) - 3ABD(\frac{y^2}{Nu}e^{-Nuy}-y^3+BD(\frac{y^3}{Nu}e^{-Nuy}-y^4) - \frac{B^2}{12}(A^7+5A^6+9A^5+25A^4+A^4-\frac{29}{5}A^3+37A^2-12A-\frac{9}{5}) + \frac{B^2}{12}(\frac{1}{5}-2) + 3A^2BD(\frac{1}{Nu}e^{-Nu}-1) - 3ABD(\frac{1}{Nu}e^{-Nu}-1) - BD(\frac{1}{Nu}e^{-Nu}+1) - NuT_1 \quad (52)$$

Resolving equations (52) into two difference equations to give:

$$\frac{\partial T_1}{\partial y} = -\frac{B^2}{12}(A^7y + 5A^6y^2 + 3A^5y^3 + 6A^5y^2 + 6A^4y^2 + A^4y^4 + 18A^4y^3 - \frac{39}{5}A^3y^5 + 2A^3y^2 + 18A^3y^3 - 18A^3y^4 + \frac{7}{5}A^2y^6 + 6A^2y^3 + 18A^2y^4 + 6A^2y^5 - 6Ay^4 - 6Ay^5) - \frac{B^2}{12}(\frac{y^8}{5} - 2y^2) + 3A^2BD(\frac{y}{Nu}e^{-Nuy} + y^2) - 3ABD(\frac{y^2}{Nu}e^{-Nuy} - y^3 - 3ABD(\frac{y^2}{Nu}e^{-Nuy} - y^3) + BD(\frac{y^3}{Nu}e^{-Nuy} - y^4) - \frac{B^2}{12}(A^7 + 5A^6 + 9A^5 + 25A^4 + A^4 - \frac{29}{5}A^3 + 37A^2 - 12A - \frac{9}{5}) + \frac{B^2}{12}(\frac{1}{5} - 2) + 3A^2BD(\frac{1}{Nu}e^{-Nu} - 1) - 3ABD(\frac{1}{Nu}e^{-Nu} - 1) - BD(\frac{1}{Nu}e^{-Nu} + 1) \quad (53A)$$

$$\frac{\partial T_1}{\partial y} = -NuT_1 \quad (53B)$$

Integrating equation (53A) to give:

$$T_1 = -\frac{B^2}{12}(\frac{A^7}{2}y^2 + \frac{5A^6}{3}y^3 + \frac{3A^2}{4}y^4 + \frac{6A^4}{3}y^3 + \frac{A^4}{5}y^5 - \frac{186A^4}{4}y^4 - \frac{39A^4}{6}y^6 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4 + \frac{18A^3}{5}y^5 + \frac{7A^2}{35}y^7 + \frac{6A^2}{4}y^4 + \frac{18A^2}{5}y^5 + \frac{6A^2}{6}y^6 - \frac{6A}{5}y^5 - \frac{6A}{6}y^6) - 3ABD(-\frac{y^2}{Nu^2}e^{-Nuy} - \frac{2y}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy}\ \frac{1}{4}y^4) - 3A^2BD\left(-\frac{y}{Nu^2}e^{-Nuy} - \frac{1}{3}y^3\right) - BD(-\frac{y^3}{Nu^2}e^{-Nuy} - \frac{3y^2}{Nu^3}e^{-Nuy} - \frac{2y}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} + \frac{1}{5}y^5) + \frac{B^2}{12}(A^7y + 5A^6y + 9A^5y + 25A^4y + A^4y - \frac{29}{5}A^3y + 37A^2y - 12Ay - \frac{9}{5}y) - 3A^2BD(\frac{y}{Nu}e^{-Nu} - y) - 3ABD(\frac{y}{Nu}e^{-Nu} - y) - BD(\frac{y}{Nu}e^{-Nu} + y) \quad (54)$$

Integrating equation (47B) to give:

$$\log T_1 = -Nuy + E \quad (55a)$$

$$T_1 = F\,e^{-Nuy} \quad (55b)$$

Adding equations (47Ai) and (47Bi) to give:

$$T_1 = -\frac{B^2}{12}(\frac{A^7}{2}y^2 + \frac{5A^6}{3}y^3 + \frac{3A^2}{4}y^4 + \frac{6A^5}{3}y^3 + \frac{6A^4}{3}y^3 + \frac{A^4}{5}y^5 - \frac{186A^4}{4}y^4 - \frac{39A^4}{6}y^6 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4 + \frac{18A^3}{5}y^5 + \frac{7A^2}{35}y^7 + \frac{6A^2}{4}y^4 + \frac{18A^2}{5}y^5 + \frac{6A^2}{6}y^6 - \frac{6A}{5}y^5 - \frac{6A}{6}y^6) - 3ABD(-\frac{y^2}{Nu^2}e^{-Nuy} - \frac{2y}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} - \frac{1}{4}y^4) - 3A^2BD\left(-\frac{y}{Nu^2}e^{-Nuy} - \frac{y}{Nu^3}e^{-Nuy} - \frac{1}{3}y^3\right) - BD(-\frac{y^3}{Nu^2}e^{-Nuy} - \frac{3y^2}{Nu^3}e^{-Nuy} - \frac{2y}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} + \frac{1}{5}y^5) + \frac{B^2}{12}(A^7y + 5A + 9A^5y + 25A^4y + A^4y - \frac{29}{5}A^3y + 37A^2y - 12Ay - \frac{9}{5}y) - 3A^2BD(\frac{y}{Nu}e^{-Nu} - y) - 3ABD(\frac{y}{Nu}e^{-Nu} - y) - BD(\frac{y}{Nu}e^{-Nu} + y) + F\,e^{-Nuy} + E \quad (56a)$$

Applying 1nitial conditions, $T_1 = 0$ at y=0 to determine E;

$$E = \frac{3A^2BD}{Nu^3} - \frac{3ABD}{Nu^3} - F \quad (56b)$$

Putting E into equation (56a), we have:

$$T_1 = -\frac{B^2}{12}(\frac{A^7}{2}y^2 + \frac{5A^6}{3}y^3 + \frac{3A^2}{4}y^4 + \frac{6A^5}{3}y^3 + \frac{6A^4}{3}y^3 + \frac{A^4}{5}y^5 - \frac{186A^4}{4}y^4 - \frac{39A^4}{6}y^6 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4 + \frac{18A^3}{5}y^5 + \frac{7A^2}{35}y^7 + \frac{6A^2}{4}y^4 + \frac{18A^2}{5}y^5 + \frac{6A^2}{6}y^6 - \frac{6A}{5}y^5 - \frac{6A}{6}y^6) - 3ABD(-\frac{y^2}{Nu^2}e^{-Nuy} - \frac{2y}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} - \frac{1}{4}y^4) - 3A^2BD\left(-\frac{y}{Nu^2}e^{-Nuy} - \frac{y}{Nu^3}e^{-Nuy} - \frac{1}{3}y^3\right) - BD(-\frac{y^3}{Nu^2}e^{-Nuy} - \frac{3y^2}{Nu^3}e^{-Nuy} - \frac{2y}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} + \frac{1}{5}y^5) + \frac{B^2}{12}(A^7y + 5A^6y + 9A^5y + 25A^4y + A^4y - \frac{29}{5}A^3y + 37A^2y - 12Ay - \frac{9}{5}y) - 3A^2BD(\frac{y}{Nu}e^{-Nu} - y) - 3ABD(\frac{y}{Nu}e^{-Nu} - y) - BD(\frac{y}{Nu}e^{-Nu} + y) + F\,e^{-Nuy} + \frac{3A^2BD}{Nu^3} - \frac{3ABD}{Nu^3} - F \quad (57)$$

Therefore, final temperature profile is obtained by combining equations (45a) and (57) thus:

$$T = T_0 + \varphi T_1$$

$$T = \frac{B}{12}(-A^4 + 6A^2y^2 - 12A^2y - 4Ay^3 + 12Ay + y^4 - 4y) + (De^{-Nuy} + 1) - (D + 1 - \varphi\frac{B^2}{12}(\frac{A^7}{2}y^2 + \frac{5A^6}{3}y^3 + \frac{3A^2}{4}y^4 + \frac{6A^5}{3}y^3 + \frac{6A^4}{3}y^3 + \frac{A^4}{5}y^5 + \frac{186A^4}{4}y^4 - \frac{39A^4}{6}y^6 + \frac{2A^3}{3}y^3 + \frac{18A^3}{4}y^4$$

$$+\frac{2A^3}{3}y^3+\frac{18A^3}{4}y^4+\frac{18A^3}{5}y^5+\frac{7A^2}{35}y^7+\frac{6A^2}{4}y^4+\frac{18A^2}{5}y^5+\frac{6A^2}{6}y^6-\frac{6A}{5}y^5-\frac{6A}{6}y^6)$$
$$-3\,\varphi ABD(-\frac{y^2}{Nu^2}e^{-Nuy}-\frac{2y}{Nu^3}e^{-Nuy}-\frac{1}{Nu^3}e^{-Nuy}-\frac{1}{4}y^4$$
$$-3\varphi A^2BD\left(-\frac{y}{Nu^2}e^{-Nuy}-\frac{y}{Nu^3}e^{-Nuy}-\frac{1}{3}y^3\right)-BD(-\frac{y^3}{Nu^2}e^{-Nuy}-\frac{3y^2}{Nu^3}e^{-Nuy}$$
$$-\frac{2y}{Nu^3}e^{-Nuy}-\frac{1}{Nu^3}e^{-Nuy}+\frac{1}{5}y^5)+\varphi\frac{B^2}{12}(A^7y+5A^6y+9A^5y+25A^4y+A^4y-\frac{29}{5}A^3y$$
$$+37A^2y-12Ay-\frac{9}{5}y)-3\varphi A^2BD(\frac{y}{Nu}e^{-Nu}-y)-3ABD(\frac{y}{Nu}e^{-Nu}-y)-\varphi BD(\frac{y}{Nu}e^{-Nu}+y)$$
$$+\varphi\,(F\,e^{-Nuy}+\frac{3A^2BD}{Nu^3}-\frac{3ABD}{Nu^3}-F) \quad (58)$$

Similarly, the velocity profile can be solved by combining equation (25B) and (23) thus:

$$\frac{\partial u}{\partial y}=-y\,e^{\varphi T}+Ae^{\varphi T} \quad (59)$$

$$\frac{\partial u}{\partial y}=(A-y)e^{\varphi T} \quad (60B)$$

$$u = u_0+\varphi\,u_1 \quad (60A)$$

Substituting for u in equation (23A) we have:

$$\frac{\partial\,u0(y)+\varphi\,u1(x,y)}{\partial y}=(A-y)e^{\varphi T} \quad (61)$$

Using Taylor series expansion, we have:

$$\frac{\partial\,u0(y)+\varphi\,u1(x,y)}{\partial y}=(A-y)[1+\frac{\varphi T}{1!}+\frac{(\varphi T)^2}{2!}+\cdots] \quad (62)$$

$$\frac{\partial u_0}{\partial y}+\varphi\frac{\partial u_1}{\partial y}=(A-y)+\varphi T_0(A-y) \quad (63)$$

The leading order and φ terms are:

$$\frac{\partial u_0}{\partial y}=(A-y) \quad (64)$$

$$\frac{\partial u_1}{\partial y}=T_0\,(A-y) \quad (65)$$

Equations (56) and (57) can be solved subject to the following conditions:

$$u_0=u_1=0,\ \text{at } y=0 \quad (66)$$

Integrating equation (56) with respect to y we have:

$$u_0=\frac{y}{2}(2-y)+J \quad (67)$$

Applying initials conditions: $u_0 = 0$ at $y = 0$ in equation (59) to give

$$J=0$$

$$u_0=\frac{y}{2}(2-y) \quad (68)$$

Equation (60) gives Newtonian velocity profile. A similar expression for equation (60) may be obtained in Meyers et-al.,(2006)

Substituting T_0 into equation (57), we have:

$$\frac{\partial u_1}{\partial y}=[\frac{B}{12}(-A^4+6A^2y^2-12A^2y-4Ay^3+12Ay+y^4-4y)+(De^{-Nuy}+1)-(D+1)](A-y) \quad (69)$$

Simplifying equation (3.61) to give:

$$\frac{\partial u_1}{\partial y}=\frac{B}{12}(-A^5+6A^3y^2-12A^3y-4A^2y^3+12A^2y+A\,y^4-4Ay)+A\,(De^{-Nuy}+1)-A(D+1) \quad (70)$$

Integrating equation (62) with respect to y we have:

$$u_1=\frac{B}{12}(-A^5y+2A^3y^3-6A^3y^2-\frac{5}{2}A^2y^4+6A^2y^2+A\,y^5-2Ay^2+A^4y^2+4A^2y^3+4Ay^3$$
$$-\frac{1}{6}y^6+\frac{4}{3}y^3)+D\left(\frac{A}{Nu}e^{-Nuy}-\frac{y^2}{2}-Ay\right)+y\,(A-1)+\frac{1}{Nu}e^{-Nuy}(y-\frac{1}{Nu})+J \quad (71a)$$

Applying initial conditions: $u_1 = 0$ at $y = 0$ to determine J

$$J=-AD\frac{1}{Nu}+\frac{1}{Nu^2}e^{-Nuy} \quad (71b)$$

Substituting J into equation (63) we have:

$$u_1=\frac{B}{12}(-A^5y+2A^3y^3-6A^3y^2-\frac{5}{2}A^2y^4+6A^2y^2+A\,y^5-2Ay^2+A^4y^2+4A^2y^3+4Ay^3-\frac{1}{6}y^6$$
$$+\frac{4}{3}y^3)+D\left(\frac{A}{Nu}e^{-Nuy}-\frac{y^2}{2}-Ay\right)+y\,(A-1)+\frac{1}{Nu}e^{-Nuy}(y-\frac{1}{Nu})-AD\frac{1}{Nu}+\frac{1}{Nu^2}e^{-Nuy} \quad (72)$$

Therefore, final velocity profile is obtained by combining equations (60) and (63) thus:

$$u = u_0+\varphi\,u_1$$
$$u=\frac{y}{2}(2-y)+\varphi\frac{B}{12}(-A^5y+2A^3y^3-6A^3y^2\frac{5}{2}A^2y^4+6A^2y^2+A\,y^5-2Ay^2+A^4y^2+4A^2y^3$$
$$+4Ay^3-\frac{1}{6}y^6+\frac{4}{3}y^3)+\varphi\,D\left(\frac{A}{Nu}e^{-Nuy}-\frac{y^2}{2}-Ay\right)+y\,\varphi\,(A-1)+\varphi\frac{1}{Nu}e^{-Nuy}(y-\frac{1}{Nu})$$
$$+\varphi\frac{1}{Nu}e^{-Nuy}(y-AD) \quad (73)$$

To find all the arbitrary constant A,C,D,E,F, and J, we apply initial conditions: to give:

$$D = -1,\ F = C = J = 0,\ B = \frac{Gr}{1+R_s}\varphi$$

$$A = -\frac{1}{Nu}e^{-Nuy} \text{ and } E = \frac{3\,Gr}{1+R_s}\frac{1}{NU^4}e^{-Nuy}\left(y - \frac{1}{Nu}\right) \quad (74)$$

Substituting equation (66) into equation (50) to give a final solution of temp., profile thus;

$$T = \frac{Gr}{1+R_s}\Big(\frac{1}{Nu^4}e^{-4Nuy} + \frac{6y^2}{Nu^2}e^{-2Nuy} - \frac{12y}{Nu^2}e^{-2Nuy} + \frac{4y}{Nu^3}e^{-3Nuy} + \frac{12y}{Nu}e^{-Nuy}$$

$$+ y^4 - 4y\Big) + (e^{-Nuy} + 1) - \varphi\frac{B^2}{12}\Big(-\frac{y^2}{Nu^7}e^{-7Nuy} + \frac{5y^2}{3Nu^6}e^{-6Nuy} - \frac{3y^4}{4Nu^2}e^{-5Nuy} - \frac{6y^3}{3Nu^5}$$

$$+\frac{6y^3}{3Nu^4}e^{-4Nuy} + \frac{y^5}{5Nu^4}e^{-4Nuy} + \frac{18y^4}{4Nu^4}e^{-4Nuy} + \frac{39y^6}{6Nu^3}e^{-3Nuy} - \frac{2y^3}{3Nu^3}e^{-3Nuy} - \frac{18y^4}{4Nu^3}e^{-3Nuy}$$

$$+\frac{18y^5}{5Nu^3}e^{-3Nuy} + \frac{7y^7}{35Nu^2}e^{-2Nuy} + \frac{6y^4}{4Nu^2}e^{-2Nuy} + \frac{18y^5}{5Nu^2}e^{-2Nuy} + \frac{y^6}{Nu^2}e^{-2Nuy} + \frac{6y^5}{5Nu}e^{-Nuy} + \frac{y^6}{Nu}e^{-Nuy}\Big) - \varphi\frac{B^2}{12}\Big(\frac{y^9}{45} - \frac{y^6}{3}\Big) -$$
$$\varphi\frac{3\,Gr}{1+R_s}\frac{1}{Nu^2}e^{-2Nuy}\Big(-\frac{y}{Nu^2}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} - \frac{1}{3}y^3\Big) - \varphi\frac{3\,Gr}{1+R_s}\frac{1}{Nu}e^{-Nuy}\Big(-\frac{y^2}{Nu^2}e^{-Nuy} - \frac{2y^2}{Nu^3}e^{-Nuy} - \frac{1}{Nu^3}e^{-Nuy} -$$
$$\frac{1}{4}y^4\Big) - \varphi\frac{Gr}{1+R_s}\Big(-\frac{y^3}{Nu^2}e^{-Nuy}$$

$$-\frac{3y^2}{Nu^3}e^{-Nuy} - \frac{2y^2}{Nu^3}e^{-Nuy} + \frac{1}{5}y^5\Big) + \varphi\frac{3\,Gr}{1+R_s}\Big(-\frac{y}{Nu^7}e^{-7Nuy} - \frac{5y}{Nu^6}e^{-6Nuy} - \frac{9y}{Nu^5}e^{-5Nuy}$$
$$-\frac{25y}{Nu^4}e^{-4Nuy} - \frac{29y}{5Nu^3}e^{-3Nuy} + \frac{37y}{Nu^2}e^{-2Nuy} - \frac{12y}{Nu}e^{-Nuy} - \frac{9y}{5}\Big) - \varphi\frac{3\,Gr}{1+R_s}\frac{1}{Nu^2}e^{-2Nuy}\Big(\frac{y}{Nu}e^{-Nu} - y\Big)$$

$$-\varphi\frac{3\,Gr}{1+R_s}\frac{1}{Nu}e^{-2Nuy}\Big(\frac{y}{Nu}e^{-Nu} - y\Big) + \varphi\frac{Gr}{1+R_s}\Big(\frac{y}{Nu}e^{-Nu} + y\Big) + \varphi\frac{3\,Gr}{1+R_s}\frac{1}{Nu^4}e^{-Nuy}\Big(\frac{1}{Nu}e^{-Nu} + 1\Big) \quad (75)$$

Substituting equation (66) into equation (50) to give final solution of velocity profile thus:

$$u = \frac{y}{2}\Big(\frac{y}{Nu}e^{-Nu} - y\Big) + \varphi\frac{Gr}{12(1+R_s)}\Big(-\frac{y}{Nu^5}e^{-5Nuy} + \frac{2y^3}{Nu^3}e^{-3Nuy} - \frac{6y^2}{Nu^2}e^{-2Nuy}$$

$$-\frac{5y^4}{2Nu^4}e^{-4Nuy} - \frac{6y^2}{Nu^2}e^{-2Nuy} + \frac{y^5}{Nu}e^{-Nuy} - \frac{2y^2}{Nu}e^{-Nuy} + \frac{4y^3}{Nu^2}e^{-2Nuy}$$

$$-\frac{4y^3}{Nu}e^{-Nuy} + \frac{y}{Nu}e^{-Nuy} - \frac{y^6}{6} + \frac{4y^3}{Nu}\Big) + \varphi\Big(\frac{1}{Nu^2}e^{-2Nuy} - \frac{y^2}{2} - \frac{y}{Nu}e^{-Nuy}\Big) + \varphi\, y\Big(\frac{1}{Nu}e^{-Nu} - 1\Big) + \varphi\frac{1}{Nu}e^{-Nuy}\Big(y - \frac{1}{Nu}\Big) + \varphi$$
$$\Big(\frac{1}{Nu^2}e^{-Nuy} + \frac{1}{Nu}e^{-Nuy}\Big) \quad (76)$$

5. DISCUSSION OF RESULTS

In this paper, the effect of the thermal radiation and the variable viscosity fluid flow an inclined plane with a free surface is investigated. The system of differential equations (26a) and (25) are meant for the velocity and temperature profile which are solved analytically using the asymptotic method of solution. The results are presented in figures 1 to 8, for velocity and temperature for various values of the flow governing parameters such as Nusselt number and Grashof number and radiation parameter and coefficient of viscosity variation.

For the analytical validation of our results, we carefully chose values for the parameters used for plotting the graphs. The following values were assumed for each graph thus;

In figure 2 and 3; (Nu =0.5, R_s= 0.2 Gr = 2.0) at various values of (φ =0.1, 0.2, 0.3) for the effect of G_r on u_y. and φ on u_y.

In figure 3; (Nu = 0.5, φ = 0.1 Gr = 5.0) at various values of (φ =0.1, 0.2, 0.3) for the effect of R_s on u_y.

In figure 4; (φ = 0.1, R_s = 0.1, G_r =5.0) at various values of (R_s= 0.1, 0.5, 1.0) for the effect of Nu on $u_{(y)}$.

In figure 5; (Nu =1.0, φ = 0.1, R_s = 0.1) at various values of (Gr =0.1, 1.0, 2.0) for the effect of Gr on $T_{(y)}$.

In figure 6: (Nu =5.0, Rs = 0.2, Gr =1.0) at various values of (φ =0.5, 1.0, 2.0) for the effect of φ on $T_{(y)}$.

In figure 7; (Nu = 1.0, φ = 0.1, Gr =2.0) at various values of (Rs = 0.1, 0.3,0.5) for the effect of Rs on $T_{(y)}$.

In figure 8; (φ = 0.1, Rs = 0.1, Gr = 1.0) at various values of (Nu =2.0, 3.0, 5.0)for the effect of Nu on $T_{(y)}$

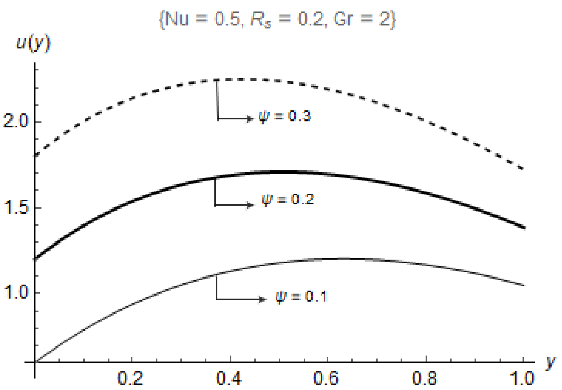


Fig. 1. Effect of Gr on u(y)

Figure 1 above shows the real application of the velocity profile for the various values of φ (0.1 - 0.3). It is observed that as the coefficient of viscosity increases the velocity also increases. An increase in Grashof number simply indicates that the fluid heat up faster and the fluid of viscosity drops and aids the flow of the fluid along an inclined plane. As a result, the velocity of the fluid increases significantly in the direction of the flow that is Newton's second law of motion is obeyed. Asibor et-al; (2017) investigated variable thermal conductivity on Jeffery fluid past a vertical porous plate with heat and mass fluxes, using an implicit finite difference method of Crank-Nicolson type. Their results on the effect of Grashof number on the velocity of the fluid indicate that the Grashof number increases when the velocity increased which are following the present result in the graph above.

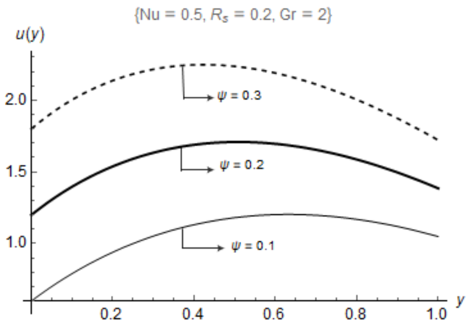


Fig. 2. Effect of ϕ on u(y)

Figure 2, shows the relationship between the viscosity and velocity of the fluid. It is observed that as viscosity increases the velocity of the fluid also increases. This occurs due to the less resistance flow of the fluid and the velocity of the fluid increases to the maximum speed at the free surface. Meanwhile**, Tshela, (**2013) investigates the flow variable viscosity fluid down an inclined plane with a free surface, using Runge Kuta Method and his results indicate that the velocity of the fluid increases as the coefficient of viscosity variation increases which is similar to the recent results of the study from the graph above.

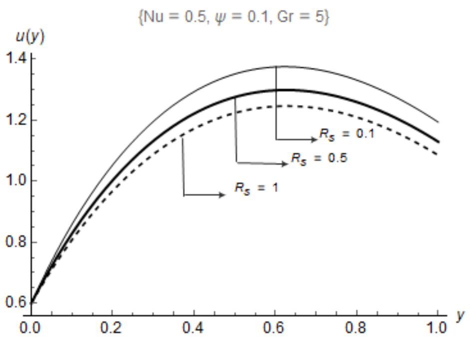


Fig. 3. Effect of R_s on u(y)

Figure 3, depicts the effect of solar radiation over the velocity of the fluid. It is observed that as the value of R_s increases the velocity of the fluid increases. The velocity increases exponentially with an increase in temperature due to the heat radiation absorbed by the fluid surface. Figure 4, shows the effect of the Nusselt number over the velocity of the fluid. It is observed that an increase in the Nusselt number decreases the velocity of the fluid. This is due to the decrease in the heat transferred to the surrounding atmosphere. The fluid regains the viscous force and the reduction in the velocity of the fluid is then observed.

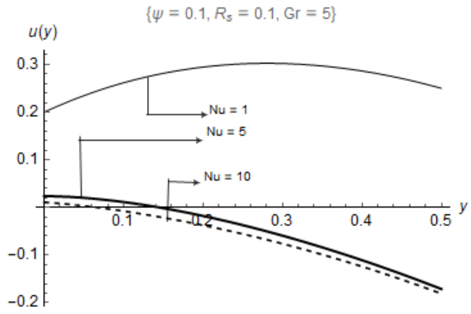


Fig. 4. Effect of Nu on u(y)

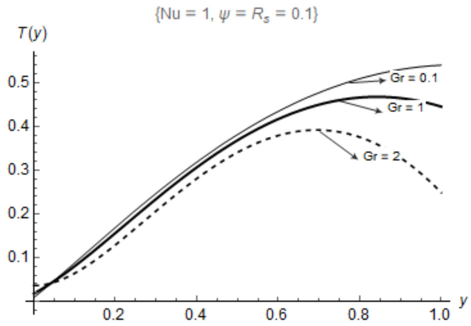


Fig. 5. Effect of Gr on T(y)

Figure 5, this shows the effect of Grashof number over the temperature of the fluid. It is observed that the Grashof number increases while the temperature of the fluid decreases. The result indicated that there is a reduction in the viscous force of the fluid due to the heat dissipation of the fluid to the atmosphere. Hence the temperature of the fluid decreases.

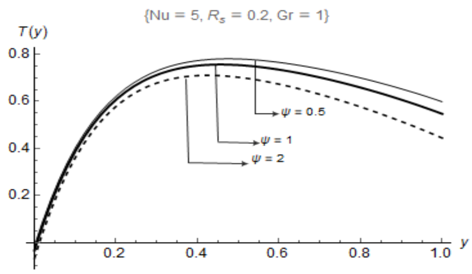
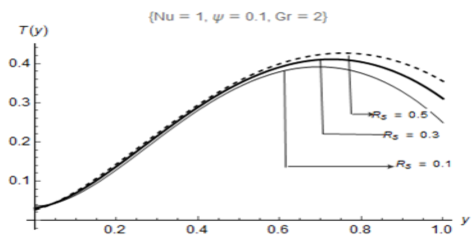


Fig. 6. Effect of ϕ on T(y)

Figure 6**,** shows the effect of viscosity on the temperature of the fluid. It is observed that viscosity increases while the temperature decreases. When ϕ increases, the temperature of the fluid decreases due to the heat lost by the internal frictional forces caused by the collision of the fluid particles.



e

Fig. 7. Effect of R_s on T(y)

Figure 7, the effect of solar radiation on temperature is depicted. It is observed that the values of the radiation parameter increase while the temperature of the fluid increases. Since heat gain by the fluid through radiation is directly proportional to the fluid which led to an increase in temperature as observed from the graph above. Asibor et-al; (2017) investigated variable thermal conductivity on Jeffery fluid past a vertical porous plate with heat and mass fluxes, using an implicit finite difference method of Crank-Nicolson type. Their results on the effect of thermal radiation on temperature of the fluid indicate that the temperature profile increases in the presence of heat generation and radiation parameter increased which is following the present result in the graph above.

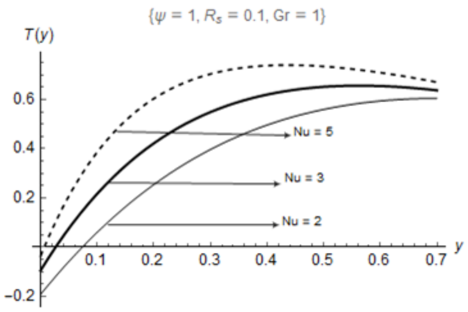


Fig. 8. Effect of Nu on T(y)

Figure 8, shows the effect of Nusselt on the temperature of the fluid. It is observed that the Nusselt number decreases while the fluid temperature increases. The temperature difference between the fluid and wall channel decreases, and the Nusselt number over the wall decreases. Therefore, an increase in temperature is observed. This result is similar to the finding of Tshela, (2013) where he analyzed the effect of Biot number on the temperature of the fluid from his investigation on the flow of a variable viscosity fluid down an inclined plane with a free surface. His results indicate the temperature of the fluid increases as the Biot number decreases.

4. CONCLUSION

This analytical study has been carried out for the effect of thermal radiation and variable viscosity on a fluid flow along an inclined plane with a free surface. The governing partial differential equations are solved analytically by the asymptotic method. The effects of velocity, temperature, and radiation parameters studied. The effects of Grashof number, Nusselt number and viscosity variation on velocity and temperature profile are shown graphically. The velocity profile of various effects of thermophysical parameters is displayed in figures 1 to 4. The general observation is that the maximum flow speed is noticed at the centerline of the flow channel. In figures 1 and 2, the rate of flow speed increases to the rise in Gr and $\varphi$and reduction due to an increase in Rs and Nu in figures 3 and 4 respectively. However, the temperature distributions of the flow and heat transfer are displayed in figures 5 to 8. Figures 5 and 6 showed that the rising values of Gr and φ bring about a reduction in temperature while the reverse is noticed in figures 7 and 8 where an increase in Rs and Nu thereby make the temperature rise **as** well.

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NOMENCLATURE

Nu Nusselt number

Gr Grashof number

C_p Heat capacity (JKg−1K−1)

Ec Eckert number

g Acceleration due to gravity (m/s^2)

h Channel height (m)

k Thermal conductivity (Wm−1K−1)

L Channel length (m)

P Pressure scale (Pa)

p Pressure (Pa)

Pe P´eclet number

Pr Prandtl number

Re Renolds number

t Time (s)

T Temperature (°C)

T_l Lower temperature

T_s Surface temperature (°C)

ΔT Temperature drop (°C)

U Velocity scale (m/s)

(u, v): Cartesian velocity (m/s)

(x, y): Cartesian coordinates (m)

ε Aspect ratio of the flow

μ Kinematic viscosity (kg/ms)

μ_0 Kinematic viscosity references (kg/ms)

Φ: Viscous dissipation function

ρ Fluid density (Kgm−3)

φ Coefficient of viscosity variation (K^{-1}).

R_s Solar radiation

β Coefficient of volume expansion (K^{-1})

ϕ Coefficient of viscosity