Total Head Evaluation using Exact and Finite Element Solutions of Laplace Equation for Seepage of Water under Sheet Pile

A R T I C L E  I N F O

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1. INTRODUCTION

Sheet pile is one of the retaining structures that is mainly affected by the seepage of water underneath its base. The seepage of the water may alter the value and the distribution of the passive pressure where failure could happen at the base of the sheet pile due to the extreme difference in hydraulic head between the up and downstream [1].

When an excavating is complete in a huge area, the soil failure due to seepage in the front of the sheet pile is a problem in two dimensions where such problem was first addressed by Terzaghi [2].

A simple relation was proposed by Henry Darcy in (1856) using a laboratory experiment that is known by Darcy’s law. Henry Darcy was the first to work in evaluating the amount of seepage in a homogenous soil and finally the Laplace partial differential equation of flow is obtained [3]. The Laplace flow equation was solved using exact solution depending on the method of separation of variables. Later on and after a half of century of advancement, finite element method (FEM) has used as a very powerful method for numerical computations in both engineering and science.

In the FEM, isoparametric elements are mapped elements that play a very important role in such numerical evaluation [4]. In numerical simulation, the determinant of
1- Develop a computer program in MATLAB to apply both exact and finite element solutions.
2- Investigate the contours of total head distribution over the studied area for both exact and finite element solutions.
3- Study the absolute error of the total head at every single node.

2. MATHEMATICAL BACKGROUND AND METHODS

2.1. Darcy Law

Darcy law can be given as below:

\[ \mathbf{v} = -\frac{k}{\mu} \left( \text{grad}[p] + \rho \mathbf{g} \right) \]  \hspace{1cm} (1)

For incompressible fluid:

\[ \text{div} \left[ \mathbf{v} \right] = 0, \text{then} \]

\[ \text{div} \left[ \mathbf{v} \right] = \text{div} \left[ -\frac{k}{\mu} \left( \text{grad}[p] + \rho \mathbf{g} \right) \right] = 0 \]  \hspace{1cm} (2)

2.2. [S] or Strong Form

Use Eq. (2) to derive [S] problem:

\[- \text{div} \left[ \frac{k}{\mu} \text{grad}[p] \right] = - \text{div} \left[ \rho \mathbf{g} \right] \]  \hspace{1cm} (3)

\[ \mathbf{v} \cdot \mathbf{n} = \mathbf{v}_o(x) \text{ on } \Gamma^N \]

\[ p(x) = p_o(x) \text{ on } \Gamma^D \]

\[ \Gamma^N \cup \Gamma^D = \partial \Omega \]

\[ \Gamma^N \cap \Gamma^D = \emptyset \]

where

\[ \mathbf{n} \text{ unit normal vector.} \]

\[ p_o \text{ prescribed values of pressure.} \]

\[ p \text{ prescribed values of pressure on the boundary.} \]

\[ \Gamma^N \text{ Neumann boundary condition.} \]

\[ \Gamma^D \text{ Dirichlet boundary condition.} \]

2.3. [V] or Weak form

Multiply both sides by \(w\) (test function) then integrate over the area (using Galerkin’s formalism):

\[ \int_B w \cdot \text{div} \left[ \frac{k}{\mu} \text{grad}[p] \right] d\Omega = \int_B w \cdot \text{div} \left[ \rho \mathbf{g} \right] d\Omega, \forall w \]  \hspace{1cm} (4)

Apply Green identity to the left side:

\[ \int_B \text{div} \left[ w \cdot \frac{k}{\mu} \text{grad}[p] \right] d\Omega - \int_B \text{grad}[w] \cdot \text{grad}[p] d\Omega = \int_B w \cdot \text{div} \left[ \rho \mathbf{g} \right] d\Omega \]  \hspace{1cm} (5)

Apply divergence theorem to the first term in the left side:

\[ \int_{\partial \Omega} \frac{k}{\mu} \text{grad}[p] \cdot \mathbf{n} d\Gamma - \int_B \text{grad}[w] \cdot \frac{k}{\mu} \text{grad}[p] d\Omega = \int_{\partial \Omega} w \cdot \text{div} \left[ \rho \mathbf{g} \right] d\Omega \]  \hspace{1cm} (6)

Then,

\[ \int_B \text{grad}[w] \cdot \frac{k}{\mu} \text{grad}[p] d\Omega = \int_{\partial \Omega} \frac{k}{\mu} \text{grad}[p] \cdot \mathbf{n} d\Gamma - \int_{\partial \Omega} w \cdot \text{div} \left[ \rho \mathbf{g} \right] d\Omega, \forall w \]  \hspace{1cm} (7)

The first term in the right hand side can be simplified by assuming 2D problem and also take in consider:
\[ \text{grad}[p] = \gamma_w \text{grad}[h] = \gamma_w \left[ \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right] = \gamma_w [0 \ 0] = 0 \] (8)

Then, Eq. (8) becomes:

\[ \int_{\Omega} \text{grad}[w]. \frac{k}{\mu} \text{grad}(p). d\Omega = - \int_{\Omega} w \cdot \text{div}[\rho \ g] \ d\Omega, \forall w \] (9)

Eq. (9) can be written as follows:

\[ \int_{\Omega} \text{grad}[w]. \frac{k \gamma_w}{\mu} \text{grad}(h).d\Omega = - \int_{\Omega} w \cdot \text{div}[F] \ d\Omega, \forall w \] (10)

where \( h \) is the head of water. When the fluid is water, then: \( \frac{k \gamma_w}{\mu} = 1.0 \). Hence, Eq. (10) can be written:

\[ \int_{\Omega} \text{grad}[w]. \text{grad}(h).d\Omega = - \int_{\Omega} w \cdot \text{div}[F] \ d\Omega, \forall w \] (11)

Eq. (11) is used in finite element formulation:

\[
\begin{align*}
\text{LHS} &= \int_{\Omega} \text{grad}[w]. \text{grad}(h).d\Omega = \int_{\Omega} \left( B^T \Theta I \right)^T \text{vec}[w^T] . \left( B^T \Theta I \right) \text{vec}[h^T] \ d\Omega \\
&= \text{vec}[w^T] \int_{\Omega} \left( B^T \Theta I \right)^T \left( B^T \Theta I \right) \text{vec}[h^T] \ d\Omega \\
\end{align*}
\]

\[
\begin{align*}
\text{RHS} &= - \int_{\Omega} w \cdot \text{div}[F] \ d\Omega = - \int_{\Omega} \left( N \Theta I \right)^T \text{vec}[B^T] \text{vec}[F^T] \ d\Omega \\
&= - \text{vec}[w^T] \int_{\Omega} \left( N \Theta I \right)^T \text{vec}[B^T] \text{vec}[F^T] \ d\Omega
\end{align*}
\]

From Eqs. (11) and (12):

\[
\begin{align*}
\text{vec}[w^T] \text{vec}[h^T] & \text{ will be dropped from both sides. Then, the full equation can be written:} \\
\begin{bmatrix} K \end{bmatrix} \begin{bmatrix} U \end{bmatrix} &= \begin{bmatrix} F \end{bmatrix} \\
\text{where} & \\
\begin{bmatrix} K \end{bmatrix} &= \int_{\Omega} \left( B^T \Theta I \right)^T \left( B^T \Theta I \right) \ d\Omega \quad \text{(Known)} \\
\begin{bmatrix} U \end{bmatrix} &= \text{vec}[h^T] \quad \text{(Unknown)} \\
\begin{bmatrix} F \end{bmatrix} &= - \int_{\Omega} \left( N \Theta I \right)^T \text{vec}[B^T] \text{vec}[F^T] \ d\Omega \quad \text{(Known)}
\end{align*}
\]

2.4. Exact Solution of Laplace Equation

For the steady state flow in 2D homogenous, Laplace equation can be written as follows:

\[ \nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \] (15)

The idealization of the steady state with given boundary conditions are shown in Fig. 1.

For the exact solution of the Laplace equation, separation of variables can be used:

\[ h(x,y) = F(x)G(y) \] (16)

\[
\begin{align*}
\frac{1}{F} \frac{d^2 F}{dx^2} &= - \frac{1}{G} \frac{d^2 G}{dy^2} = -k
\end{align*}
\]

Then,

\[
\frac{d^2 F}{dx^2} + k F = 0
\]

From the left and right boundary conditions:

\[
F(0) = 0, F(a) = 0 \implies k = \left( \frac{\pi}{a} \right)^2
\]

(19)

\[
F(x) = F_n(x) = \sin \left( \frac{n \pi x}{a} \right)
\]

(20)

\[
G(y) = G_n(y) = A_n e^{(n \pi y / a)} + B_n e^{(-n \pi y / a)}
\]

(21)

By applying both bottom and top boundary conditions:

\[
h(x,y) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi x}{a} \right) \sinh \left( \frac{n \pi y}{a} \right)
\]

(22)

\[
A_n = \left. \frac{2}{a \sinh(n \pi b / a)} \right| \int_{0}^{a} f(x) \sin \left( \frac{n \pi x}{a} \right) dx
\]

(23)

where \( n \) is number of points in the Fourier series.

![Fig. 1. Steady state dealization of 2D-homogenous Laplace quation.](image-url)
3. RESULTS AND ANALYSIS

The description of the studied case can be shown in Fig. 2. The problem represents the seepage of water under sheet pile from upstream level to downstream level. Both exact and numerical solutions have been investigated using different Fourier points and number of nodes. The element numbering has started from the upper left side of the mesh where the node numbers are given in clock wise order.

Several trials have been investigated for the case study as shown in Table 1. All the studied trials have the same dimensions with width (a) and height (b) of 10 m. In addition, the upstream and downstream heads for all the trails are 5 m, and 1 m respectively. The variation of the head contours for both FEM and analytical solutions for the first trial (one point Fourier, nine nodes in X and Y directions) have shown in Fig. 3. In the FEM, the contours of the head distribution are concentrated on the upper left side where the high head in the upstream is located. However, in the exact solution; the contours of the head distribution are concentrated on the upper middle side in symmetrical shape.

Similarly, the variation of the head contours for both FEM and analytical solutions for the second trial (one point Fourier, 11 nodes in X and Y directions) have shown in Fig. 4. In the FEM, the contours of the head distribution are a little smoother compared to FEM in the first trail that has lower number of nodes. Likewise, the variation of the head contours for both FEM and analytical solutions for the third trial (one point Fourier, 15 nodes in X and Y directions) have shown in Fig. 5. In the FEM, the contours of the head distribution are started to be smoother compared to FEM in the previous trails that have lower number of nodes. In addition, the variation of the head contours for both FEM and analytical solutions for the fourth trial (one point Fourier, 21 nodes in X and Y directions) have shown in Fig. 6. Smoother head contours distributions are observed for FEM compared to the previous trials.

<table>
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<tr>
<th>Trial no.</th>
<th>N (Fourier points)</th>
<th>a (m)</th>
<th>b (m)</th>
<th>V1 (m)</th>
<th>V2 (m)</th>
<th>XSeed (no. nodes)</th>
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Fig. 3. Contours of head for trial 1, (a) Exact solution, and (b) FEM.

Fig. 4. Contours of head for trial 2, (a) Exact solution, and (b) FEM.

Fig. 5. The contours of head for trial 3, (a) Exact solution, and (b) FEM.

Fig. 6. The contours of head for trial 4, (a) Exact solution, and (b) FEM.
77 nodes in X and Y directions) have shown in Fig. 10. In the FEM, the contours of the head distribution are concentrated on the upper left side where the high head in the upstream is located. In the exact solution, as the Fourier Points increased from 1 to 5; the contours of the head distribution are moved towards the region of high head in the upstream zone on the upper left side. The variation of the head contours for both FEM and analytical solutions for the ninth trial (five points Fourier; 99 nodes in X and Y directions) have shown in Fig. 11. As the numbers of nodes are increased in FEM, smoother head contours distributions are noticed. The variation of the head contours for both FEM and analytical solutions for the tenth trial (five points Fourier; 101 nodes in X and Y directions) have shown in Fig. 12. As the numbers of nodes are increased in FEM, smoother head contours distributions are noticed.

Fig. 5. Contours of head for trial 3, (a) Exact solution, and (b) FEM.

Fig. 6. Contours of head for trial 4, (a) Exact solution, and (b) FEM.

The variation of the total head with node number for both exact and FEM solutions using five nodes as total number of nodes in x and y directions is shown in Fig. 13. The FEM solution is very close to the exact solution in all the nodes with the 100% value prediction in the locations that has zero total head.

The variation of the absolute error of total head prediction with node number for both exact and FEM solutions using five nodes as total number of nodes in x and y directions is shown in Fig. 14. In most of the nodes, the FEM solution is very close to the exact solution with almost zero absolute error. However, in several nodes; there is a quite high absolute error prediction with the maximum value of 61% at node 22 (upper right side of the mesh).
The variation of the total head with node number for both exact and FEM solutions using nine nodes as total number of nodes in x and y directions is shown in Fig. 15. With increasing the number of nodes, the FEM solution is very close to the exact solution in most of the nodes with the 100% value prediction in the locations that has zero total head.

The variation of the absolute error of total head prediction with node number for both exact and FEM solutions using nine nodes as total number of nodes in x and y directions is shown in Fig. 16. In most of the nodes, the FEM solution is very close to the exact solution with almost zero absolute error. Nevertheless, in several nodes; there is a quite high absolute error prediction with the maximum value of 61% at node 75 (upper right zone of the downstream).

![Fig. 9. Contours of head for trial 7, (a) Exact solution, and (b) FEM.](image_1)

![Fig. 10. Contours of head for trial 8, (a) Exact solution, and (b) FEM.](image_2)

![Fig. 11. Contours of head for trial 9, (a) Exact solution, and (b) FEM.](image_3)

![Fig. 12. Contours of head for trial 10, (a) Exact solution, and (b) FEM.](image_4)

![Fig. 13. Variation of total head with node number for both exact and FEM solutions (number of nodes in X and Y directions = 5).](image_5)
number of nodes in $x$ and $y$ directions is shown in Fig. 17. With increasing the number of nodes, the FEM solution is very close to the exact solution in most of the nodes with the 100% value prediction in the locations that has zero total head.

![Fig. 14. Variation of absolute error with node number for total head (number of nodes in $X$ and $Y$ directions = 5).](image1)

![Fig. 15. Variation of total head with node number for both exact and fem solutions (number of nodes in $X$ and $Y$ directions = 9).](image2)

![Fig. 16. Variation of absolute error with node number for total head (number of nodes in $X$ and $Y$ directions = 9).](image3)

![Fig. 17. Variation of total head with node number for both exact and fem solutions (number of nodes in $X$ and $Y$ directions = 11).](image4)

The variation of the absolute error of total head prediction with node number for both exact and FEM solutions using 11 nodes as total number of nodes in x and y directions is shown in Fig. 18. In most of the nodes, the FEM solution is very close to the exact solution with almost zero absolute error. Nevertheless, in several nodes; there is a quite high absolute error prediction with the maximum value of 88% at node 113 (upper right zone of the downstream).

![Fig. 18. Variation of absolute error with node number for total head (number of nodes in $X$ and $Y$ directions = 11).](image5)

4. CONCLUSIONS

In this study, an exact solution approach as Fourier series of the Laplace partial differential equation has been introduced to solve one of the geotechnical problems, the seepage of water under a sheet pile, which adopts also a finite element method. Even though of considering Fourier series an exact solution for the Laplace equation, it is originally an approximate procedure due to n points which cannot be infinite. The following results can be pointed out:

1. Both exact and finite element solutions have a good match in distribution over the studied area especially when the Fourier points are increased.
2. In general, the absolute errors of total head prediction are reasonable except in the top surface where the absolute error may reach 88% when the total number of nodes are 11 in both x and y directions.

REFERENCES


